Geometric and spectral properties of locally tessellating planar graphs

Matthias Keller [*]
TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz, Germany

Norbert Peyerimhoff [†]
Department of Math. Sciences
University of Durham
Durham DH1 2LE, UK

May 12, 2008

Abstract

In this article, we derive bounds for values of the global geometry of locally tessellating planar graphs, namely, the Cheeger constant and exponential growth, in terms of combinatorial curvatures. We also discuss spectral implications for the Laplacians.

1 Introduction

A locally tessellating planar graph $\mathcal{G}$ is a tiling of the plane with all faces to be polygons with finitely or infinitely many boundary edges (see Subsection 2.1 for precise definitions). The edges of $\mathcal{G}$ are continuous rectifiable curves without self-intersections. Faces with infinitely many boundary edges are called infinigons and occur, e.g., in the case of planar trees. The sets of vertices, edges and faces of $\mathcal{G}$ are denoted by $\mathcal{V}, \mathcal{E}$ and $\mathcal{F}$. $d(v,w)$ denotes the combinatorial distance between two vertices $v, w \in \mathcal{V}$, where each edge is assumed to have combinatorial length one.

Useful *local* concepts of the graph $\mathcal{G}$ are combinatorial curvature notions. The finest curvature notion is defined on the corners of $\mathcal{G}$. A corner is a pair $(v,f) \in \mathcal{V} \times \mathcal{F}$, where v is a vertex of the face f. The set of all corners is denoted by $\mathcal{C}$. The *corner curvature* κ_C is then defined as

$$\kappa_C(v,f) = \frac{1}{|v|} + \frac{1}{|f|} - \frac{1}{2},$$

where $|v|$ and $|f|$ denote the degree of the vertex v and the face f. If f is an infinigon, we set $|f| = \infty$ and $1/|f| = 0$. The *curvature at a vertex* $v \in \mathcal{V}$ is given by the sum

$$\kappa(v) = \sum_{(v,f) \in \mathcal{C}} \kappa_C(v,f) = 1 - \frac{|v|}{2} + \sum_{f : v \in f} \frac{1}{|f|}.$$

[*]e-mail: matthias.keller@mathematik.tu-chemnitz.de
[†]e-mail: norbert.peyerimhoff@durham.ac.uk

For a finite set $W \subset \mathcal{V}$ we define $\kappa(W) = \sum_{v \in W} \kappa(v)$. These combinatorial curvature definitions arise naturally from considerations of the Euler characteristic and tessellations of closed surfaces, and they allow to prove a combinatorial Gauß-Bonnet formula (see [BP1, Thm 1.4]). Similar combinatorial curvature notions have been introduced by many other authors, e.g., [Gro, Hi, St, Woe].

The aim of this paper is to establish connections between local curvature conditions and characteristic values of the *global geometry* of the graph $\mathcal{G}$, in particular the exponential growth and Cheeger constants. For a finite subset $W \subset \mathcal{V}$, let $\mathrm{vol}(W) = \sum_{v \in W} |v|$. The exponential growth is defined as follows (note that the value $\mu(\mathcal{G})$ does not depend on the choice of center $v \in \mathcal{V}$):

Definition 1. *The* exponential growth $\mu(\mathcal{G})$ *is given by*

$$\mu(\mathcal{G}) = \limsup_{n \to \infty} \frac{\log \mathrm{vol}(B_n(v))}{n},$$

where $B_n(v) = \{w \in \mathcal{V} \mid d(v,w) \leq n\}$ *denotes the (combinatorial) ball of radius* n *about* v.

We also consider the following two types of Cheeger constants.

Definition 2. *Let*

$$\alpha(\mathcal{G}) = \inf_{\substack{W \subseteq \mathcal{V}, \\ |W| < \infty}} \frac{|\partial_E W|}{|W|} \quad \text{and} \quad \widetilde{\alpha}(\mathcal{G}) = \inf_{\substack{W \subseteq \mathcal{V}, \\ |W| < \infty}} \frac{|\partial_E W|}{\mathrm{vol}(W)}$$

where $\partial_E W$ *is the set of all edges* $e \in \mathcal{E}$ *connecting a vertex in* W *with a vertex in* $\mathcal{V} \backslash W$. $\alpha(\mathcal{G})$ *is called the* physical Cheeger constant *and* $\widetilde{\alpha}(\mathcal{G})$ *the* combinatorial Cheeger constant *of the graph* $\mathcal{G}$.

The attributes *physical* and *combinatorial* in the previous definition are motivated by the fact that these Cheeger constants are closely linked to two types of Laplacians: The *physical Laplacian* is used frequently in the community of Mathematical Physicists and is defined as follows:

$$(\Delta \varphi)(v) = |v| \varphi(v) - \sum_{w \sim v} \varphi(w). \tag{1}$$

Note that Δ is an unbounded operator if there is no bound on the vertex degree of $\mathcal{G}$. The *combinatorial Laplacian* $\widetilde{\Delta}$ is a bounded operator and appears in the context of spectral geometry (see, e.g., [DKa, DKe, Woe2]):

$$(\widetilde{\Delta} \varphi)(v) = \varphi(v) - \frac{1}{|v|} \sum_{u \sim v} \varphi(u). \tag{2}$$

Both operators are defined in and are self-adjoint with respect to different l^2-spaces (see Subsection 2.2). In the case of fixed vertex degree, both operators are multiples of each other.

Our main geometric results are given in Subsection 2.3, where we

- provide lower bounds for both Cheeger constants in terms of combinatorial curvatures (see Theorem 1 below),

- provide upper bounds for the exponential growth in terms of an upper vertex bound (see Theorem 2 below).

Even though Theorem 2(b) is formulated in terms of bounds on vertex and face degrees, it can also be considered as an estimate in terms of combinatorial curvature, as is explained in the remark following the theorem. In fact, the proof is based on the corresponding curvature version.

Now we discuss connections to the spectrum. The Cheeger constant and the exponential growth were first introduced in the context of Riemannian manifolds and were useful invariants to estimate the bottom of the (essential) spectrum of the Laplacian (see [Che] and [Br]). An analogous inequality between the Cheeger constant and the bottom of the spectrum in the discrete case of graphs was first proved by [Do] and [Al]. This inequality is also useful in the study of expander graphs. [Al] noted also the connection between this inequality and the Max Flow-Min Cut Theorem (see also [Chu] and [Gri]). For other connections between isoperimetric inequalities and lower bounds of eigenvalues in both continuous and discrete settings see, e.g., [CGY].

The best results about the relations between the combinatorial Cheeger constant, the exponential growth, and the bottom $\widetilde{\lambda}_0(\mathcal{G})$ and $\widetilde{\lambda}_0^{ess}(\mathcal{G})$ of the (essential) spectrum of the *combinatorial Laplacians* $\widetilde{\Delta}$ are due to K. Fujiwara (see [Fu1] and [Fu2]):

$$1 - \sqrt{1 - \widetilde{\alpha}^2(\mathcal{G})} \le \widetilde{\lambda}_0(\mathcal{G}) \le \widetilde{\lambda}_0^{ess}(\mathcal{G}) \le 1 - \frac{2e^{\mu(\mathcal{G})/2}}{1 + e^{\mu(\mathcal{G})}}. \tag{3}$$

These estimates are sharp in the case of regular trees. Using these estimates and Theorems 1 and 2, we obtain

- lower and upper estimates on the bottom of the (essential) spectrum of the combinatorial Laplacian in terms of combinatorial curvature (see Corollaries 1 and 2).

Since there are estimates to compare the bottom of the (essential) spectrum of the combinatorial Laplacian with the physical Laplacian (see for instance [Ke]) these results can be also formulated for the physical Laplacian.

A lower estimate for the bottom of the essential spectrum of the combinatorial Laplacian via the combinatorial Cheeger constant at infinity can be found in [Fu2, Cor. 3]. This yields a discrete analogue for the combinatorial Laplacian of the result in [DL] about the *emptiness of the essential spectrum* for complete simply connected manifolds with curvature converging to minus infinity. Corresponding results about the emptiness of the essential spectrum for the physical Laplacian can be found in [Ke, Woj].

Finally, let us discuss two other interesting types of eigenfunctions, namely, *strictly positive eigenfunctions* and *finitely supported eigenfunctions*, and illustrate all concepts in two examples.

For the discrete case of a graph, it was shown in [DKa, Prop. 1.5] that the equation $\widetilde{\Delta} f = \lambda f$ has a *positive solution* if and only if $\lambda \le \widetilde{\lambda}_0(\mathcal{G})$. This characterisation of the bottom of the spectrum was well known before in the context of Riemannian manifolds (see, e.g., [Sull] and the references therein). In the reverse direction, this characterisation might be used in concrete cases to determine the bottom of the spectrum of an infinite graph.

On the other hand, *finitely supported solutions* of the equation $\widetilde{\Delta} f = \lambda f$ are obviously l^2-eigenfunctions and, therefore, they can only exist for eigenvalues $\lambda \geq \widetilde{\lambda}_0(\mathcal{G})$. Existence of finitely supported eigenfunctions in Penrose tilings was first observed in [KS]. Their existence is a purely discrete phenomenon, since in the case of a non-compact, connected Riemannian manifold the eigenvalue equation $\Delta f = \lambda f$ cannot have compactly supported eigenfunctions (a fact which is known as the *unique continuation principle*; see [Ar]). These finitely supported eigenfunctions coincide with the discontinuities of the integrated density of states (or spectral density function). See, e.g., the articles [KLS, LV] and the references therein for more details about this connection.

Examples. *(a) We consider the periodic tessellation $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ in Figure 1. We assume that all edges are straight Euclidean segments of length one.*

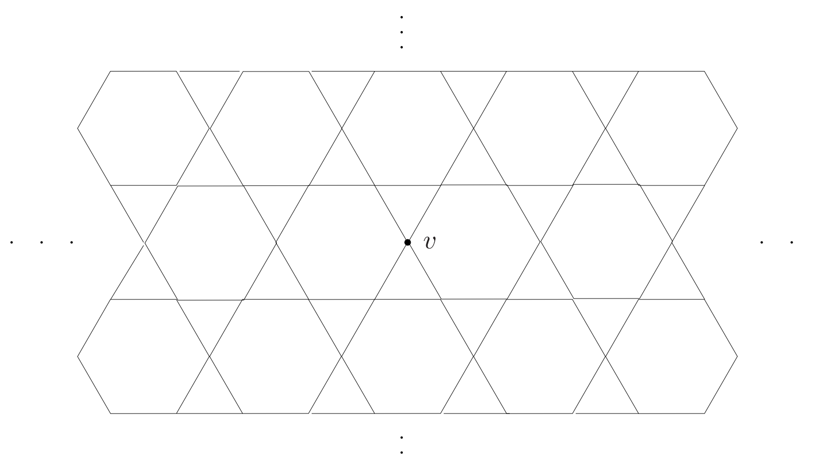

Figure 1: Plane tessellation with regular triangles and hexagons

We first show that $\mu(\mathcal{G}) = 0$: Choose a fixed radius $0 < r < 1/2$. Then all Euclidean balls of radius r centered at all vertices in $\mathcal{V}$ are pairwise disjoint. On the other hand, the vertices in the combinatorial ball $B_n(v)$ are contained in the Euclidean ball of radius n, centered at v. Both facts together imply that combinatorial balls grow only polynomially and the exponential growth is zero. As a consequence, this graph cannot contain a binary tree as a subgraph. Moreover, using (3)*, we conclude that*

$$\widetilde{\lambda}_0(\mathcal{G}) = \widetilde{\lambda}_0^{ess}(\mathcal{G}) = 0 \quad \text{and} \quad \widetilde{\alpha}(\mathcal{G}) = \alpha(\mathcal{G}) = 0.$$

Finally, $\mathcal{G}$ does admit finitely supported eigenfunctions, namely, choose $p \in \mathbb{R}^2$ to be the center of a hexagon and define $f(p + e^{2\pi i/6}) = (-1)^i$ (i.e., choose alternating values $1, -1, 1, -1, 1, -1$ clockwise around the vertices of the hexagon) and $f(v) = 0$ for all other vertices. Then we have $\widetilde{\Delta} f = \frac{3}{2} f$.

(b) Let $\mathcal{T}_p$ denote the p-regular tree. In this case, spectrum and essential spectrum of the combinatorial Laplacian coincide and are given by the interval (see, e.g., [Sun, App. 3])

$$\left[1 - \frac{2\sqrt{p-1}}{p}, 1 + \frac{2\sqrt{p-1}}{p}\right].$$

Consequently, $\widetilde{\Delta} f = \lambda f$ admits a positive solution if and only if $\lambda \leq 1 - 2\sqrt{p-1}/p$. Moreover, we have $\widetilde{\alpha}(\mathcal{T}_p) = \frac{p-2}{p}$, $\alpha(\mathcal{T}_p) = p-2$ and $\mu(\mathcal{T}_p) = \log(p-1)$. Note that a regular tree doesn't admit l^2-eigenfunctions. For otherwise, we could choose a vertex v at which our eigenfunction doesn't vanish and take its radialisation with respect to this vertex. This radialisation would be again a non-vanishing l^2-eigenfunction with the same eigenvalue and, since its values would only depend on the distance to v, there would be an easy recursion formula for its values. The precise form of the recursion formula would then contradict to the requirement that the function lies in l^2.

Acknowledgements. Matthias Keller would like to thank Daniel Lenz who encouraged him to study the connection between curvature and spectral theory. Matthias Keller was supported during this work by the German Business Foundation (sdw).

2 Basic notions and main results

In the first two subsections, we provide the notions which haven't yet been introduced in full detail in the Introduction. In Subsections 2.3 and 2.4, we state our main results.

2.1 Locally tessellating planar graphs

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a planar graph (with $\mathcal{V}$ and $\mathcal{E}$ the set of vertices and edges) embedded in $\mathbb{R}^2$. The faces f of $\mathcal{G}$ are the closures of the connected components in $\mathbb{R}^2 \setminus \bigcup_{e \in E} e$. The set of faces is denoted by $\mathcal{F}$.

We further assume that $\mathcal{G}$ has no loops, no multiple edges and no vertices of degree one (terminal vertices). We write $e = vw$, if the edge e connects the vertices v, w. Moreover, we assume that every vertex has finite degree and that every bounded open set in $\mathbb{R}^2$ meets only finitely many faces of $\mathcal{G}$. We call a planar graph with these properties *simple*. The *boundary of a face* f is the subgraph $\partial f = (\mathcal{V} \cap f, \mathcal{E} \cap f)$. We call a sequence of edges $e_1, \ldots, e_n$ a *walk of length* n if there is a corresponding sequence of vertices $v_1, \ldots, v_{n+1}$ such that $e_i = v_i v_{i+1}$. A walk is called a *path* if there is no repetition in the corresponding sequence of vertices $v_1, \ldots, v_n$.

A simple planar graph $\mathcal{G}$ is called a *locally tessellating planar graph* if the following additional conditions are satisfied:

i.) Any edge is contained in precisely two different faces.

ii.) Any two faces are either disjoint or have precisely a vertex or a path of edges in common. In the case that the length of the path is greater then one, then both faces are unbounded.

iii.) Any face is homeomorphic to the closure of an open disc $\mathbb{D} \subset \mathbb{R}^2$, to $\mathbb{R}^2 \setminus \mathbb{D}$ or to the upper half plane $\mathbb{R} \times \mathbb{R}_+ \subset \mathbb{R}^2$ and its boundary is a path.

Note that these properties force the graph $\mathcal{G}$ to be connected. Examples are tessellations $\mathbb{R}^2$ introduced in [BP1, BP2], trees in $\mathbb{R}^2$, and particular finite tessellations on the sphere mapped to $\mathbb{R}^2$ via stereographic projection.

When we consider the vertex degree as a function on $\mathcal{V}$ we write $\deg(v) = |v|$ for $v \in \mathcal{V}$. Moreover we define the degree $|f|$ of a face $f \in F$ to be the length of the shortest closed walk in the subgraph ∂f meeting all its vertices. If there is no such finite walk we set $|f| = \infty$. $v \sim w$ means that $d(v, w) = 1$, i.e., v and w are neighbors. A (finite or infinite) path with associated vertex sequence $\dots v_i v_{i+1} v_{i+2} \dots$ is called a *geodesic*, if we have $d(v_i, v_j) = |i - j|$ for all pairs of vertices in the path.

2.2 Laplacians

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a locally tessellating planar graph. The operators Δ and $\widetilde{\Delta}$ were already introduced in (1) and (2). They are symmetric operators and initially defined on the space

$$c_c(\mathcal{V}) := \{\varphi : \mathcal{V} \to \mathbb{R} \mid |\mathrm{supp}\ \varphi| < \infty\}$$

of functions with finite support. However, they have unique self-adjoint extensions on different l^2-spaces: Let $g : \mathcal{V} \to (0, \infty)$ be a weight function on the vertices of the graph $\mathcal{G}$ and

$$l^2(\mathcal{V}, g) := \{\varphi : \mathcal{V} \to \mathbb{R} \mid \langle \varphi, \varphi \rangle_g := \sum_{v \in V} g(v)|\varphi(v)|^2 < \infty\}.$$

For $g = 1$ we simply write $l^2(\mathcal{V})$.

Then the combinatorial Laplacian can be extended to a bounded self-adjoint operator on all of $l^2(\mathcal{V}, \deg)$. The physical Laplacian has also a unique self-adjoint extension in the space $l^2(\mathcal{V})$ (see [We] or [Woj]). Note, however, that the adjacency operator need not be essentially self adjoint (see [MW, Section 3] and the references therein). We denote the self-adjoint extensions of both Laplacians, again, by $\widetilde{\Delta}$ and Δ.

Furthermore, we define the restriction of the combinatorial Laplacian on the complement of a finite set K of vertices. Let $P_K : l^2(\mathcal{V}, \deg) \to l^2(\mathcal{V} \setminus K, \deg)$ be the canonical projection and $i_K : l^2(\mathcal{V} \setminus K, \deg) \to l^2(\mathcal{V}, \deg)$ be its dual operator, which is the continuation by 0 on K. We write $\widetilde{\Delta}_K = P_K \widetilde{\Delta} i_K$. Of particular importance is the bottom of the spectrum $\widetilde{\lambda}_0(\mathcal{G})$ and of the essential spectrum $\widetilde{\lambda}_0^{ess}(\mathcal{G})$. $\widetilde{\lambda}_0(\mathcal{G})$ can be characterised as the infimum of the Rayleight-Ritz quotient over all non-zero functions $f \in l^2(\mathcal{V}, \deg)$, i.e.,

$$\widetilde{\lambda}_0(\mathcal{G}) = \inf \left\{ \frac{\langle \widetilde{\Delta} f, f \rangle_{\deg}}{\langle f, f \rangle_{\deg}} : f \neq 0, f \in l^2(\mathcal{V}, \deg) \right\}.$$

Similarly, $\widetilde{\lambda}_0^{ess}(\mathcal{G})$ can be obtained via

$$\widetilde{\lambda}_0^{ess}(\mathcal{G}) = \lim_{n \to \infty} \inf \left\{ \frac{\langle \widetilde{\Delta}_{B_n} f, f \rangle_{\deg}}{\langle f, f \rangle_{\deg}} : f \neq 0, f \in l^2(\mathcal{V} \backslash B_n, \deg) \right\}, \qquad (4)$$

where B_n are balls of radius n around any fixed vertex $v \in \mathcal{V}$. A proof of (4) can be found in [Ke]. Obviously, we have $\widetilde{\lambda}_0(\mathcal{G}) \leq \widetilde{\lambda}_0^{ess}(\mathcal{G})$. Equality holds in the following case:

Proposition 1. *Assume that there is a subgroup Γ of the automorphism group of $\mathcal{G}$ with $\sup_{\gamma\in\Gamma} d(v,\gamma v) = \infty$ for some vertex $v \in \mathcal{V}$. Then we have*

$$\widetilde{\lambda}_0(\mathcal{G}) = \widetilde{\lambda}_0^{ess}(\mathcal{G}).$$

Proof. For the bottom of the spectrum not to lie in the essential spectrum would mean that it is an isolated eigenvalue of finite multiplicity. But this cannot be the case (see Fact 1 in [Sun, p. 259]). □

Analogous statements hold for the bottom of the (essential) spectrum of the physical Laplacian.

2.3 Cheeger constant and exponential growth estimates

The physical and combinatorial Cheeger constants were introduced in Definition 2. It is easy to see that they are linked to the physical and combinatorial Laplacians via the equations:

$$\alpha(\mathcal{G}) = \inf_{\substack{W\subseteq\mathcal{V},\\ |W|<\infty}} \frac{\langle \Delta\chi_W, \chi_W\rangle}{\langle \chi_W, \chi_W\rangle} \quad \text{and} \quad \widetilde{\alpha}(\mathcal{G}) = \inf_{\substack{W\subseteq\mathcal{V},\\ |W|<\infty}} \frac{\langle \widetilde{\Delta}\chi_W, \chi_W\rangle_{\deg}}{\langle \chi_W, \chi_W\rangle_{\deg}},$$

where χ_W denotes the characteristic function of the set $W \subseteq \mathcal{V}$. Note, in particular, that the combinatorial Cheeger constant is always bounded from above by $\widetilde{\alpha}(\mathcal{G}) \leq 1$.

Next, we state the Cheeger constant estimates:

Theorem 1. *Let $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{F})$ be a locally tessellating planar graph and $3 \leq q \leq \infty\}$ such that $|f| \leq q$ for all faces $f \in F$.*

(a) For some $a > 0$, let $\kappa(v) \leq -a$ for all $v \in \mathcal{V}$. Then we have

$$\alpha(\mathcal{G}) \geq \frac{2q}{q-2}a.$$

(b) For some $c > 0$, let $\frac{1}{|v|}\kappa(v) \leq -c$ for all $v \in \mathcal{V}$. Then we have

$$\widetilde{\alpha}(\mathcal{G}) \geq \frac{2q}{q-2}c.$$

Moreover, the above estimates are sharp in the case of regular trees. (Note that in the case $q = \infty$ we set $\frac{2q}{q-2} = 2$.)

Remark. *The combinatorial Cheeger constant of all non-positively curved* regular *plane tessellation $\mathcal{G}_{p,q}$ (with all vertices satisfying $|v| = p$ and faces satisfying $|f| = q$) was explicitly calculated in [HJL] and [HiShi] as*

$$\widetilde{\alpha}(\mathcal{G}_{p,q}) = \frac{p-2}{p}\sqrt{1 - \frac{4}{(p-2)(q-2)}}.$$

Our estimate gives in this case

$$\widetilde{\alpha}(\mathcal{G}_{p,q}) \geq \frac{(p-2)(q-2)-4}{p(q-2)}.$$

Before considering the exponential growth of a locally tessellating planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$, let us first introduce the *cut locus* $\mathrm{Cut}(v)$ of a vertex $v \in \mathcal{V}$. $\mathrm{Cut}(v)$ denotes the set of all vertices w, at which $d_v := d(v, \cdot)$ attains a local maximum, i.e., we have $w \in \mathrm{Cut}(v)$ if $d_v(w') \leq d_v(w)$ for all $w' \sim w$. $\mathcal{G}$ is *without cut locus* if $\mathrm{Cut}(v) = \emptyset$ for all $v \in \mathcal{V}$. Obviously, the cut locus of a finite graph is never empty. It was proved in [BP2, Thm. 1] that plane tessellations with everywhere non-positive corner curvature are graphs without cut locus. Moreover, let $\mathcal{T}_p$ denote the regular tree with $|v| = p$ for all vertices.

Theorem 2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a locally tessellating planar graph without cut locus.*

(a) If there exists $p \geq 3$ such that

$$|v| \leq p \quad \forall\, v \in \mathcal{V}, \tag{5}$$

then we have

$$\mu(\mathcal{G}) \leq \mu(\mathcal{T}_p) = \log(p-1).$$

(b) If there exist $p \geq 3$ such that (5) is satisfied and $q \in \{3,4,6\}$ such that

$$|f| = q \quad \forall\, f \in \mathcal{F},$$

(i.e., $\mathcal{G}$ is face-regular) then we have

$$\mu(\mathcal{G}) \leq \mu(\mathcal{G}_{p,q}) = \log\left(\frac{p}{2} - \frac{2}{q-2} + \sqrt{\left(\frac{p}{2} - \frac{2}{q-2}\right)^2 - 1}\right).$$

Remark. *For the reader's convenience, Theorem 2(b) was stated in "more familiar" terms of vertex and face degrees. However, the statement has an equivalent reformulation in terms of curvature: Let $\mathcal{G}$ be a locally tessellating planar graph without cut locus satisfying $|f| = q$ for all faces and $q \in \{3,4,6\}$. For some $b \geq 0$, let $-b \leq \kappa(v)$ for all $v \in \mathcal{V}$. Then we have*

$$\mu(\mathcal{G}) \leq \log(\tau + \sqrt{\tau^2 - 1}),$$

where $\tau = 1 + \frac{q}{q-2} b \geq 1$. The inequality is sharp (with the optimal choice of b) in the case of regular graphs $\mathcal{G}_{p,q}$. In fact, the proof will be given for this equivalent reformulation. (Note that the constants p and b in the two formulations are related by $b = \frac{q-2}{q} p - 1$.)

Since the regular plane tessellations $\mathcal{G}_{p,q}$ can be considered as combinatorial analogues of *constant curvature space forms* in Riemannian geometry, it is natural to conjecture the following discrete version of a *Bishop volume comparison result* (see, e.g., [GaHuLa, Theorem 3.101] for the case of a Riemannian manifold).

Conjecture. *Let $p, q \geq 3$ with $1/p + 1/q \leq 1/2$ be given. Then we have*

$$\mu(\mathcal{G}) \leq \mu(\mathcal{G}_{p,q}), \tag{6}$$

for all locally tessellating planar graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ without cut locus satisfying $|v| \leq p$, $|f| \leq q$.

Theorem 2 confirms this conjecture for the cases $q = 3$ and $q = \infty$. However, it seems difficult to prove this seemingly obvious estimate (6) for general face degree bounds $q \geq 3$. Assuming the above conjecture to be true, the comparison of the exponential growth of a locally tessellating planar graph with upper vertex degree bound p and of the regular tree $\mathcal{T}_p$, as given in Theorem 2(a), is quite good if all faces of $\mathcal{G}$ satisfy $|f| \geq 6$. For example, we have in the case $(p,q) = (5,6)$:

$$1.307\cdots = \log(2+\sqrt{3}) = \mu(\mathcal{G}_{5,6}) \leq \mu(\mathcal{T}_5) = \log 4 = 1.381\ldots.$$

An direct consequence of [BP1, Corollary 5.2] is the following lower bound for the exponential growth:

Theorem 3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a locally tessellating planar graph without cut locus and $a > 0$ such that $\kappa(v) \leq -a$ for all vertices $v \in \mathcal{V}$. Assume there is $3 \leq q \leq \infty$ such that we have $|f| \leq q$ for all faces $f \in \mathcal{F}$. Then we have*

$$\mu(G) \geq \log\left(1 + \frac{2q}{q-1}a\right).$$

Moreover, this estimate is sharp in the case of regular trees. (In the case $q = \infty$, we set $\frac{2q}{q-1} = 2$.)

We like to finish this subsection by a few additional useful facts: Let

$$S_n(v) = \{w \in \mathcal{V} \mid d(v,w) = n\}$$

be the (combinatorial) sphere of radius n about $v \in \mathcal{V}$. If there is a uniform upper bound on the vertex degree and if $s_n := |S_n(v)|$ is a non-decreasing sequence, one easily checks that

$$\mu(\mathcal{G}) = \limsup_{n\to\infty} \frac{\log s_n}{n}. \tag{7}$$

Yet another Cheeger constant $h(\mathcal{G})$ was considered in [BS]:

$$h(\mathcal{G}) = \inf_{\substack{W \subseteq \mathcal{V}, \\ |W| < \infty}} \frac{|\partial_V W|}{|W|},$$

where $\partial_V W$ is the set of all vertices $v \in \mathcal{V}\backslash W$ which are end points of an edge in $\partial_E W$. In the case that $\mu(\mathcal{G})$ is presented by (7), this Cheeger constant is related to the exponential growth by

$$e^{\mu(\mathcal{G})} \geq 1 + h(\mathcal{G}),$$

with equality in the case of regular trees.

2.4 Spectral applications

An immediate consequence of Fujiwara's lower estimate (3) and Theorem 1 is the following *combinatorial analogue of McKean's Theorem* (see [McK] for the case of a Riemannian manifold):

Corollary 1 (Combinatorial version of McKean's Theorem)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a locally tessellating planar graph and $3 \leq q \leq \infty$ such that $|f| \leq q$ for all faces $f \in F$. For some $c > 0$, let $\frac{1}{|v|}\kappa(v) \leq -c$ for all $v \in \mathcal{V}$. Then we have*

$$1 - \sqrt{1 - \left(\frac{2q}{q-2}c\right)^2} \leq \widetilde{\lambda}_0(\mathcal{G}).$$

This estimate is sharp in the case of regular trees.

Combining Theorem 2(a), the curvature version of Theorem 2(b) (see the remark of the theorem) and Fujiwara's upper estimate (3), we obtain:

Corollary 2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a locally tessellating planar graph without cut locus.*

(a) If there exists $p \geq 3$ such that

$$|v| \leq p \quad \forall\, v \in \mathcal{V}, \tag{8}$$

then we have

$$\widetilde{\lambda}_0^{ess}(\mathcal{G}) \leq \widetilde{\lambda}_0^{ess}(\mathcal{T}_p) = 1 - \frac{2\sqrt{p-1}}{p}.$$

(b) If there exist $q \in \{3, 4, 6\}$ with $|f| = q$ for all $f \in \mathcal{F}$, and $b > 0$ with $-b \leq \kappa(v)$ for all $v \in \mathcal{V}$, then we have

$$\widetilde{\lambda}_0^{ess}(\mathcal{G}) \leq 1 - \frac{2\sqrt{\tau + \sqrt{\tau^2 - 1}}}{1 + \tau + \sqrt{\tau^2 - 1}},$$

where $\tau = 1 + \frac{q}{q-2}b$.

Next we indicate implications of the above results for the spectrum of the *physical Laplacian*. Let $\lambda_0(\mathcal{G})$ and $\lambda_)^{ess}(\mathcal{G})$ denote the bottom of the (essential) spectrum of the physical Laplacian Δ and, for $n \geq 0$, let

$$m_n = \inf_{w \in V \setminus B_{n-1}(v)} |w| \quad \text{and} \quad M_n = \sup_{w \in V \setminus B_{n-1}(v)} |w|,$$

where $v \in \mathcal{V}$ is an arbitrary vertex and $B_{-1}(v) = \emptyset$. Moreover let $m_\infty = \lim_{n\to\infty} m_n$ and $M_\infty = \lim_{n\to\infty} M_n$. Then we have, by [Do]

$$\lambda_0(\mathcal{G}) \geq \frac{\alpha(\mathcal{G})^2}{2M} \quad \text{and} \quad \lambda_0^{ess}(\mathcal{G}) \geq \frac{\alpha_\infty(\mathcal{G})^2}{2M_\infty}, \tag{9}$$

where $\alpha_\infty(\mathcal{G})$ denotes the physical Cheeger constant at infinity, defined in [Ke]. In general we can also estimate, as demonstrated in [Ke],

$$m_0\widetilde{\lambda}_0(\mathcal{G}) \leq \lambda_0(\mathcal{G}) \leq M_0\widetilde{\lambda}_0(\mathcal{G}) \quad \text{and} \quad m_\infty\widetilde{\lambda}_0^{ess}(\mathcal{G}) \leq \lambda_0^{ess}(\mathcal{G}) \leq M_\infty\widetilde{\lambda}_0^{ess}(\mathcal{G}).$$

Via this inequalities we can estimate the bottom of the (essential) spectrum of the physical Laplacian Δ by the estimates of Corollary 1 and 2 for the combinatorial Laplacian.

Before we look at an explicit example, let us mention the following result about the absence of finitely supported eigenfunctions in the case of non-positive corner curvature:

Theorem 4 (see [KLPS, Theorem 4])**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation (in the restricted sense of [BP2]) with non-positive corner curvature in all corners. Then the combinatorial Laplacian does not admit finitely supported eigenfunctions.*

Note that Theorem 4 becomes wrong if we replace "non-positive corner curvature" by the weaker assumption "non-positive vertex curvature", since Example (a) of the Introduction is a graph with vanishing vertex curvature which admits finitely supported eigenfunctions.

Let us, finally, apply the above results in an example.

Example. *We consider the regular tessellation $\mathcal{G}_{6,6}$. Using our geometric results in this article, we obtain*

$$\widetilde{\alpha}(\mathcal{G}_{6,6}) \geq \frac{1}{2} \quad \textit{and} \quad \mu(\mathcal{G}_{6,6}) = \log \frac{1+\sqrt{21}}{2} \approx 1.5668.$$

Proposition 1 tells us that $\widetilde{\lambda}_0(\mathcal{G}_{6,6}) = \widetilde{\lambda}_0^{ess}(\mathcal{G}_{6,6})$, and with our results in this Subsection we can conclude that

$$\begin{aligned} \widetilde{\lambda}_0(\mathcal{G}_{6,6}) = \widetilde{\lambda}_0^{ess}(\mathcal{G}_{6,6}) \quad &\in \quad \left[1 - \frac{\sqrt{3}}{2}, 1 - 2\frac{\sqrt{3}+\sqrt{7}}{7+\sqrt{21}}\right] \\ &\approx \quad [0.1340, 0.2441]. \end{aligned}$$

Using the explicit formula for the Cheeger constant in [HJL] in this particular case, we obtain $\widetilde{\alpha}(\mathcal{G}_{6,6}) = \frac{1}{\sqrt{3}} \approx 0.5774$ and we can shrink this interval to

$$\begin{aligned} \widetilde{\lambda}_0(\mathcal{G}_{6,6}) = \widetilde{\lambda}_0^{ess}(\mathcal{G}_{6,6}) \quad &\in \quad \left[1 - \sqrt{\frac{2}{3}}, 1 - 2\frac{\sqrt{3}+\sqrt{7}}{7+\sqrt{21}}\right] \\ &\approx \quad [0.1835, 0.2441]. \end{aligned}$$

Note that the physical Laplacian is just a multiple of the combinatorial Laplacian ($\Delta = 6\widetilde{\Delta}$). Finally, Theorem 4 guarantees that there are no finitely supported eigenfunctions in $\mathcal{G}_{6,6}$.

3 Proof of Theorem 1

The heart of the proof of Theorem 1 is Proposition 2 below. An earlier version of this proposition in the dual setting (see [BP1, Prop. 2.1]) was originally obtained by helpful discussions with Harm Derksen. Let us first introduce some important notions related to a locally tessellating planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$.

For a finite set $W \subseteq \mathcal{V}$ let $\mathcal{G}_W = (W, \mathcal{E}_W, \mathcal{F}_W)$ be the subgraph of $\mathcal{G}$ induced by W, where $\mathcal{E}_W$ are the edges in $\mathcal{E}$ with both end points in W and $\mathcal{F}_W$ are the faces induced by the graph $(W, \mathcal{E}_W)$. Euler's formula states for a finite and connected subgraph $\mathcal{G}_W$ (observe that $\mathcal{F}_W$ contains also the unbounded face):

$$|W| - |\mathcal{E}_W| + |\mathcal{F}_W| = 2. \tag{10}$$

By $\partial_F W$, we denote the set of faces in F which contain an edge of $\partial_E W$. Moreover, we define the *inner degree* of a face $f \in \partial_F W$ by

$$|f|^i{}_W = |f \cap W|.$$

In the following, we need the two important formulas which hold for arbitrary finite and connected subgraphs $\mathcal{G}_W = (W, \mathcal{E}_W, \mathcal{F}_W)$. The first formula is easy to see and reads as

$$\sum_{v \in W} |v| = 2|\mathcal{E}_W| + |\partial_E W|. \tag{11}$$

Since W is finite, the set $\mathcal{F}_W$ contains at least one face which is not in $\mathcal{F}$, namely the unbounded face surrounding $\mathcal{G}_W$, but there can be more. Define $C(W) = |\mathcal{F}_W| - |\mathcal{F}_W \cap \mathcal{F}| \geq 1$. Note that $|\mathcal{F}_W \cap \mathcal{F}|$ is the number of faces in $\mathcal{F}$ which are entirely enclosed by edges of $\mathcal{E}_W$. Sorting the following sum over vertices according to faces gives the second formula

$$\begin{aligned} \sum_{v \in W} \sum_{f \ni v} \frac{1}{|f|} &= |\mathcal{F}_W \cap \mathcal{F}| + \sum_{f \in \partial_F W} \frac{|f|_W^i}{|f|} \\ &= |\mathcal{F}_W| - C(W) + \sum_{f \in \partial_F W} \frac{|f|_W^i}{|f|}. \end{aligned} \tag{12}$$

Proposition 2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a locally tessellating planar graph and $W \subset \mathcal{V}$ be a finite set of vertices such that the induced subgraph $\mathcal{G}_W$ is connected. Then we have*

$$\kappa(W) = 2 - C(W) - \frac{|\partial_E W|}{2} + \sum_{f \in \partial_F W} \frac{|f|_W^i}{|f|}$$

Proof. By the equations (11), (12) and (10) we conclude

$$\begin{aligned} \kappa(W) &= \sum_{v \in W} \left(1 - \frac{|v|}{2} + \sum_{f \ni v} \frac{1}{|f|} \right) \\ &= |W| - |\mathcal{E}_W| - \frac{|\partial_E W|}{2} + |\mathcal{F}_W| - C(W) + \sum_{f \in \partial_F W} \frac{|f|_W^i}{|f|} \\ &= 2 - C(W) - \frac{|\partial_E W|}{2} + \sum_{f \in \partial_F W} \frac{|f|_W^i}{|f|}. \end{aligned}$$

□

Proposition 3. *Let $G = (V, E, F)$ be a locally tessellating planar graph and $3 \leq q \leq \infty$ such that $|f| \leq q$ for $f \in F$. Let $W \subset \mathcal{V}$ be a finite set of vertices such that the induced subgraph $\mathcal{G}_W$ is connected. Then we have*

$$|\partial_E W| \geq \frac{2q}{q-2}(2 - C(W) - \kappa(W)).$$

Proof. Since $\mathcal{G}$ is locally tessellating, every edge $e \in \partial_E W$ separates precisely two different faces. The edge obtains a direction by its start vertex to be in $\mathcal{V} \backslash W$ and its end vertex to be in W. Thus it makes sense to refer to the faces at the left and right side of the edge e. Thus every edge $e \in \partial_E W$ determines a unique corner $(v, f) \in W \times \partial_F W$, where $v \in W$ is the end vertex of e and f is

the face at the left side of e. The so defined map $\partial_E W \to W \times \partial_F W$ is clearly injective, and thus we have

$$\sum_{f \in \partial_F W} |f|_W^i = |\{(v,f) \in W \times \partial_F W : v \in f\}| \geq |\partial_E W|.$$

Using this fact and $|f| \leq q$ for all $f \in \mathcal{F}$, we conclude with Proposition 2

$$2 - C(W) - \kappa(W) = \frac{|\partial_E W|}{2} - \sum_{f \in \partial_F W} \frac{|f|_W^i}{|f|} \leq |\partial_E W| \left(\frac{1}{2} - \frac{1}{q} \right),$$

which proves the inequality in the proposition. □

Note that the Cheeger constants in Definition 2 are obtained by taking the infimum of a particular expression over all finite subsets $W \subset \mathcal{V}$. In fact, we can restrict ourselves to consider only finite sets W for which the induces graph $\mathcal{G}_W$ is connected. This follows from the observation that, for a given finite set $W \subset \mathcal{V}$, we can always find a non-empty subset $W_0 \subset W$ such that $\mathcal{G}_{W_0}$ is a connected component of $\mathcal{G}_w$ and that $|\partial_E W_0|/\mathrm{vol}(W_0) \leq |\partial_e W|/\mathrm{vol}(W)$ or $|\partial_E W_0|/|W_0| \leq |\partial_e W|/|W|$, respectively. We can reduce the sets under consideration even further. Let $W \subset \mathcal{V}$ be a finite set such that $\mathcal{G}_W$ is connected. Note that $\mathcal{G}_w$ has only one unbounded face. By adding all vertices of $\mathcal{V}$ contained in the union of all bounded faces of $\mathcal{G}_w$, we obtain a bigger finite set $P_W \supset W$ such that $C(P_W) = 1$. (Note that all bounded faces of $\mathcal{G}_{P_W}$ are also faces of the original graph $\mathcal{G}$.) We call a finite set $P \subset \mathcal{V}$ with connected graph $\mathcal{G}_P$ and $C(P) = 1$ a *polygon*. Clearly, we have $|\partial_E P_W|/\mathrm{vol}(P_W) \leq |\partial_e W|/\mathrm{vol}(W)$ and $|\partial_E P_W|/|P_W| \leq |\partial_e W|/|W|$. Thus it suffices for the definition of the Cheeger constants to take the infimum only over all polygons.

With this final observation we can now prove Theorem 1.

Proof of Theorem 1. Let $W \subset \mathcal{V}$ be a polygon. Since $C(W) = 1$, we conclude from Proposition 3 that

$$\frac{|\partial_E W|}{|W|} \geq \frac{2q}{q-2} \frac{-\kappa(W)}{|W|} \geq \frac{2q}{q-2} a.$$

Taking the infimum over all polygons yields part (a) of the theorem.

For the proof of part (b), recall that $-\kappa(v) \geq c \cdot |v|$ for all vertices $v \in \mathcal{V}$. This implies that

$$\frac{-\kappa(W)}{\mathrm{vol}(W)} = \frac{-\sum_{v \in W} \kappa(v)}{\sum_{v \in W} |v|} \geq c,$$

and, consequently, for polygons $W \subset \mathcal{V}$,

$$\frac{|\partial_E W|}{\mathrm{vol}(W)} \geq \frac{2q}{q-2} \frac{-\kappa(W)}{\mathrm{vol}(W)} \geq \frac{2q}{q-2} c.$$

The statement follows now again by taking the infimum over all polygons. □

4 Proof of Theorem 2

Parts (a) and (b) of Theorem 2 have very different proofs. We present them separately.

Proof of Theorem 2 (a). We choose a vertex $v_0 \in \mathcal{V}$ and introduce the following functions $m, M : \mathcal{F} \to \{0, 1, 2, \ldots, \infty\}$:

$$\begin{aligned} m(f) &= \min\{d(w, v_0) \mid w \in \partial f\}, \\ M(f) &= \max\{d(w, v_0) \mid w \in \partial f\}. \end{aligned}$$

Note that the face f "opens up" at distance $m(f)$ and "closes up" at distance $M(f)$ from v_0. We call a face f *finite*, if $M(f) < \infty$.

The idea of the proof is to "open up" successively every finite face $f \in \mathcal{F}$ into an infinigon without violating the vertex bound. In this way, we will build up a comparison tree $\mathcal{T}$ with the same vertex bound p and satisfying $\mu(\mathcal{G}) \le \mu(\mathcal{T})$. It turns out, however, that finite faces f with more than one vertex in the sphere $S_{M(f)}(v_0)$ cause problems in this "opening up" procedure (since the distance relations to the vertex v_0 will be changed). Therefore, we first modify the tessellation $\mathcal{G}$ by removing all edges connecting two vertices v, w at the same distance to v_0. The modified planar graph is denoted by $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{F}_0)$. To keep track, we add at each of the vertices v, w a short terminal edge. These terminal edges do not belong "officially" to the graph $\mathcal{G}_0$ and serve merely as reminders that an edge can be added in their place without violating the vertex bound of the graph. Moreover, we can only guarantee $\mu(\mathcal{G}_0) \ge \mu(\mathcal{G})$, if these inofficial edges are included in $\mathcal{G}_0$. (At the end of the procedure we will replace all "inofficial" terminal edges by infinite trees rooted in v and w.) The modification $\mathcal{G} \to \mathcal{G}_0$ is illustrated in Figure 2. (For convenience, the vertices belonging to distance spheres $S_n(v_0)$ are arranged to lie on concentric Euclidean circles around v_0.)

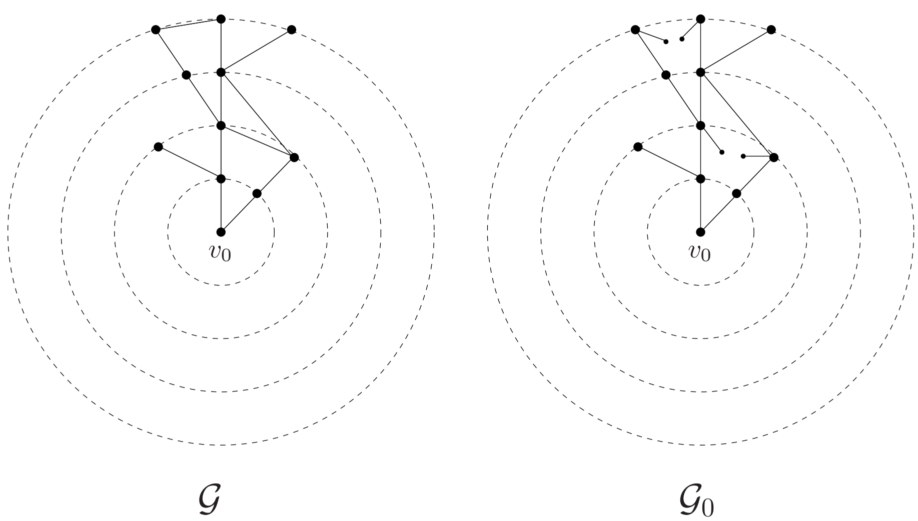

Figure 2: Removing edges between vertices on the same spheres and replacing them by "inofficial" terminal edges

Note that none of the distance relations of the vertices in $\mathcal{G}_0$ (without the inofficial terminal edges) to the vertex v_0 are changed and that we still have $\mathrm{Cut}(v_0) = \emptyset$. Moreover, the modified graph G_0 (without the inofficial terminal edges) has a new set of faces $\mathcal{F}_0$. Every finite face f of $\mathcal{G}_0$ has now even degree, since f opens up at a single vertex in the sphere $S_{m(f)}(v_0)$ and f closes up at a single vertex in the sphere $S_{M(f)}(v_0)$.

We order all finite faces $f_0, f_1, f_2, \dots$ of $\mathcal{G}_0$ such that we have

$$M(f_0) \le M(f_1) \le M(f_2) \le ...$$

Next we explain the first step of our procedure, namely, how to open up f_0 into an infinigon $\widetilde{f_0}$. Let $n = M(f_0) \ge 1$ and $w \in \partial f_0$ such that $d(w, v_0) = n$. Since $C(v_0) = \emptyset$, we can find an infinite geodesic ray $w_0 = w, w_1, w_2, \dots \in \mathcal{V}$ such that $d(w_i, v_0) = n + i$. We may think of v_0 as being the origin of the plane and of $w_0, w_1, \dots$ as being arranged to lie on the positive vertical coordinate axis at heights $n, n+1, \dots$ with straight edges between them. Now we cut our plane along this geodesic ray, i.e., replace the ray by two parallel copies of the ray and thus preventing the face f_0 from closing up at distance n. In this way, f_0 becomes an infinigon, which we denote by $\widetilde{f_0}$. (In fact, we rotationally shrink the angle 2π to $2\pi - \epsilon$ around v_0 to open up a conic sector of angle ϵ containing the infinigon $\widetilde{f_0}$.) The procedure is illustrated in Figure 3. Note that the vertices w_i are replaced by two copies $w_i^{(1)}, w_i^{(2)}$, such that $w_i^{(j)}$ is connected to $w_{i+1}^{(j)}$ for $j = 1, 2$ and $w_i^{(1)}$ inherits all previous neighbors of w_i at one side of the ray and $w_i^{(2)}$ inherits all previous neighbors of w_i at the other side of the ray (this concerns in particular also the "inofficial" vertices). In this way we obtain a new planar graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{F}_1)$.

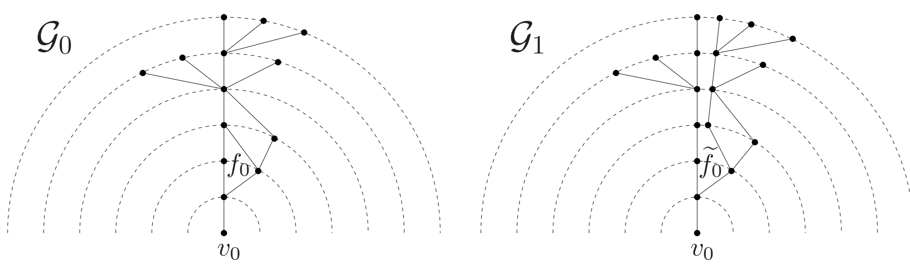

Figure 3: Changing the finite face f_0 into an infinigon $\widetilde{f_0}$

The graph $\mathcal{G}_1$ is still connected. Note also that we have

$$|w_0^{(1)}| + |w_0^{(2)}| = |w_0| + 1, \tag{13}$$
$$|w_i^{(1)}| + |w_i^{(2)}| = |w_i| + 2, \quad \forall\, i \ge 1. \tag{14}$$

After including the inofficial terminal edges in the graph $\mathcal{G}_1$, we still have

$$|v| \le p \quad \forall\, v \in \mathcal{V}_1,$$

and (13), (14) imply that $\mu(\mathcal{G}_1) \ge \mu(\mathcal{G}_0) \ge \mu(\mathcal{G})$.

In the second step we carry out the same procedure with the face $f_1 \in \mathcal{F}_1$, and obtain a new connected planar graph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{F}_2)$, with $f_1 \in \mathcal{F}_1$

replaced by the infinigon $\widetilde{f}_1 \in \mathcal{F}_2$. Again, after including the inofficial terminal edges, the graph $\mathcal{G}_2$ has vertex bound p and satisfies $\mu(\mathcal{G}_2) \geq \mu(\mathcal{G}_1) \geq \mu(\mathcal{G})$.

It is now clear how to repeat the procedure. Note that for every radius $n \geq 1$ there is a large enough $j \geq 1$ such that the graphs $\mathcal{G}_j, \mathcal{G}_{j+1}, \mathcal{G}_{j+2}, \dots$ remain unaltered inside the balls $B_n(\mathcal{G}_k, v_0)$. This fact guarantees that there is a well-defined limiting graph associated to the sequence $\mathcal{G}_j$. This limit is a connected tree $\mathcal{T}_0$ (since all faces of $\mathcal{T}_0$ are infinigons). In $\mathcal{T}_0$, we replace now finally the inofficial terminal edges by infinite trees, rooted at the corresponding proper vertices of the tree $\mathcal{T}_0$, with branching sequence $1, p-1, p-1, p-1, \dots$. These infinite trees can be nicely fitted into the infinigons to yield an infinite planar tree $\mathcal{T}$ with vertex bound p and satisfying $\mu(\mathcal{T}) \geq \mu(\mathcal{G})$. Since we obviously have $\mu(\mathcal{T}) \leq \mu(\mathcal{T}_p) = \log(p-1)$, the proof of part (a) of the theorem is finished. $\square$

Proof of Theorem 2 (b). We prove the equivalent curvature version of the statement, given in the remark after the theorem. Since $|f| = q < \infty$ for all faces f, $\mathcal{G}$ is a tessellating plane graph in the sense of [BP1] and we have

$$\kappa(v) = 1 - \frac{q-2}{q}|v|.$$

Since $\{\kappa(v) \mid v \in \mathcal{V}\}$ is a discrete set and bounded from below by $-b$, we can assume, without loss of generality, that $-b$ is of the form $1 - \frac{q-2}{q}p$, for some integer value $p \geq 3$. (In fact, p is the optimal upper bound on the vertex degree of $\mathcal{G}$.)

Let S_n, B_n be the combinatorial spheres and balls in $\mathcal{G}$ with respect to a reference vertex $v_0 \in \mathcal{V}$ and $s_n = |S_n|$. Corollary 6.4 of [BP1] states that we have

$$s_{n+1} - s_n = \frac{2q}{q-2}(1 - \kappa(B_n)).$$

Applying this equation twice, we derive

$$s_{n+2} - 2s_{n+1} + s_n = -\frac{2q}{q-2}\kappa(S_{n+1}) \leq \frac{2q}{q-2} b s_{n+1}.$$

Hence we obtain the following recursion inequality

$$s_{n+2} \leq 2\tau s_{n+1} - s_n, \quad s_1 \leq p, \ s_0 = 1,$$

with $\tau = 1 + \frac{q}{q-2}b \geq 1$. It is easy to see that the sequence

$$\sigma_{n+2} = 2\tau\sigma_{n+1} - \sigma_n, \qquad \sigma_1 = p, \ \sigma_0 = 1, \tag{15}$$

is strictly increasing and dominates the sequence s_n. Moreover, σ_n describes the cardinality of a sphere of radius n in the regular tessellation $\mathcal{G}_{p,q}$. This implies that $\mu(\mathcal{G}) \leq \mu(\mathcal{G}_{p,q})$.

Now, we return to the sequence σ_n, as defined in (15). We first consider the case $\tau > 1$. The recursion formula implies that

$$\sigma_n = u\left(\tau - \sqrt{\tau^2 - 1}\right)^n + v\left(\tau + \sqrt{\tau^2 - 1}\right)^n,$$

with constants $u, v \in \mathbb{R}$ chosen in such a way that the initial conditions are satisfied. Since

$$0 < \tau - \sqrt{\tau^2 - 1} < 1,$$

we conclude that $v \neq 0$, for otherwise we would have $\sigma_n \to 0$, contradicting to the fact that $\mathcal{G}_{p,q}$ is an infinite graph. Hence, σ_n behaves asymptotically like

$$\sigma_n \sim v \left(\tau + \sqrt{\tau^2 - 1}\right)^n,$$

with a positive constant v. This, together with (7) implies that

$$\mu(\mathcal{G}_{p,q}) = \lim_{n\to\infty} \frac{\log \sigma_n}{n} = \log\left(\tau + \sqrt{\tau^2 - 1}\right). \tag{16}$$

In the case $\tau = 1$, the sequence (15) is simply given by $\sigma_n = n(p-1)+1$. Linear growth of σ_n implies that $\mu(\mathcal{G}_{p,q}) = 0$, which also coincides with (16). □

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